Stat 206B Lecture 3 Notes

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January 23, 2018

1 Donsker's Theorem

1.1 Portmanteau theorem and Donsker's theorem

Donsker's theorem shows that Brownian motion can be viewed as a limit of random walks. Its formulation requires a change of point of view. We have $B = (B_t, t \geq 0)$ defined on $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, \text{Leb})$, and given $\omega \in [0,1]$, we get a path $(B_t(\omega), t \geq 0)$. This implies the existence of a probability measure on $(C[0,\infty), b)$, where b is the general coordinates [what is this?]; this measure is called *Wiener measure*. Wiener measure is the pushforward of Lebesgue measure under B.

We use P_0 for the Wiener measure, denoting P_x as the distribution of $x + B$. This is nice from the perspective of B as a Markov process, where we want to be able to start from any state. We need this for continuous analogues of "first step analysis of Markov chains."

To prove Donsker's theorem, we state the Portmanteau theorem here.

Theorem 1.1 (Portmanteau). *The following are equivalent:*

1. $Eg(X_n) \to Eg(X)$ *for all uniformly continuous* g.

2. $Eg(X_n) \to Eg(X)$ *when* g *is Lipschitz (constant 1).*

3. $Eg(X_n) \to Eg(X)$ *when* $g = 1_G$ *for a Borel set* $G \subseteq [0,1]$ *with* $P(X \in \partial G) = 0$ *(where* ∂G *is the boundary of* G *as a metric space).*

4. $Eg(X_n) \to Eg(X)$ *when the set* D *of discontinuities of* g *has* $P(B \in D) = 0$*, and* g *is bounded.*

Note that $X_n \xrightarrow{d} X \implies g(X_n) \xrightarrow{d} g(X)$ for all g that have $P(X \in D) = 0$ (even unbounded g).

Theorem 1.2 (Donsker). *The Wiener measure* P_0 *on* $C[0,1]$ *is the limit in distribution of rescaled random walks. More precisely, let* $S_n = X_1 + \cdots + X_n$*, with* $S_0 = 0$*, where the* X_i *are independent with* $EX_i = 0$ *and* $EX_i^2 = 1$*. Construct* $B_n(t) := S_k/\sqrt{n}$ *if* $t = k/n$ *and made continuous on* $[0,1]$ *by linear interpolation between these* $n+1$ *points.*

Proof. We can see that

1. B_n is a random element of $C[0,1]$.
2. $B_n(t)$ is a random variable in each t.
3. $t \to B_n(t)$ is continuous.

We must show that $B_n \xrightarrow{d} B$. What does this convergence mean? This is weak-* convergence[1] of the *measures*. $C[0,1]$ is a metric space with the distance $d(f,g) := \sup_{0 \le t \le 1} |f(t) - g(t)|$. By the Portmanteau theorem, it is sufficient to show that $Eg(B_n) \to Eg(B)$ for every bounded continuous $g : C[0,1] \to \mathbb{R}$.

For the rest of the proof, see Durrett or Kallenberg. $\square$

1.2 Applications of Donsker's theorem

We can get nice statements about Brownian motion by treating it as the limit of random walks.

Example 1.1. Take $g(f) := \sup_{0 \le t \le 1} f(t)$. The function g is Lipschitz continuous. We learn that

$$\sup_{0 \le t \le 1} B_n(t) \xrightarrow{d} \sup_{0 \le t \le 1} B(t),$$

as the left hand side is $g(B_n)$, and the right is $g(B)$. By construction, the left hand side is $\max_{0 \le k \le n} S_k / \sqrt{n}$. Note that

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} B(1)$$

is the Central Limit Theorem.

If $k/n \to t$ as $k = k_n \to \infty$ and $n \to \infty$, then

$$\frac{S_k}{\sqrt{n}} \xrightarrow{d} \sqrt{t} B(1) \stackrel{d}{=} B(t).$$

Donsker's theorem is often called the "invariance principle." This is because the limit does not depend on the distribution of X. This is powerful because you can compute laws of functionals of B by doing limits for *particular* random walks.

Example 1.2. It is easy to handle the maximum of a simple random walk. Recall the reflection principle for a simple random walk. Take $X_i = \pm 1$ with probability $1/2$ each. What is $P(\max_{0 \le k \le n} S_k \ge m)$? Consider a path with $0 \le S_n = b \le m$ and maximum $M_n \ge m$. We claim that

$$P(M_n \ge m, S_n \ge b) = P(S_n = 2m - b).$$

[1] Some people just call this weak convergence.

This is because we have 2^n equally likely paths and a bijection between paths with the left hand side to paths with the right hand side.

If we sum over $b < m$, we learn that

$$P(M_n \geq m, S_n < m) = P(M_n \geq m, S_n > m) = P(S_n > m)$$

These two probabilities plus $P(S_n = m)$ gives us $P(M_n \geq n)$. So

$$P(M_n \geq m) - P(S_n = m) = 2P(S_n > m),$$

which gives us

$$P(M_n \geq m) = 2P(S_n > m) + P(S_n = m) = P(|S_n| > m) + P(S_n = m)$$

Scale and let $n \to \infty$, using $|S_n/\sqrt{n}| \xrightarrow{d} |B(1)|$. This gives us

$$\frac{M_n}{\sqrt{n}} \xrightarrow{d} |B(1)|\,.$$

By Donsker's theorem, we get that

$$\sup_{0 \leq t \leq 1} B(t) \stackrel{d}{=} |B(1)|\,.$$

Example 1.3. Above, we used the reflection principle for a random walk. In the limit, this implies a reflection principle for Brownian motion. The same argument gives us

$$\left(\frac{M_n}{\sqrt{n}}, \frac{S_n}{\sqrt{n}}\right) \xrightarrow{d} (M(1), B(1))$$

Proof. Take $g(\frac{M_n}{\sqrt{n}}, \frac{S_n}{\sqrt{n}})$ for bounded continuous g. Call this $\tilde{g}(B_n)$. This is bounded and converges in expectation to g applied to the right hand side. By the Portmanteau theorem, we are done. ☐

This implies that we can evaluate the law of $(M(1), B(1))$ by random walk limits. For $-\infty < x \leq y$, we can write down

$$P(B(1) \leq x, M(1) \geq y) = P(B(1) \geq 2y - x).$$

This statement can be proved in two ways: by the above proof sketch using Donsker's theorem or by the reflection principle for Brownian motion. We state the principle here.

Theorem 1.3 (Reflection Principle[2]). *Let $T_y := \inf\{t : B_t = y\}$. Fix $y > 0$, and let*

$$\hat{B}(t) := \begin{cases} B(t), & t \leq T_y \\ y - (B_t - y) & y > T_y. \end{cases}$$

Then $\hat{B} \stackrel{d}{=} B$.

[2]This was proved by Désiré André in the 1800s. Even back then, people studied the finite dimensional distributions of Brownian motion.